Cosby's Rule

Connecting Summation Operations, Definite Integrals, and the Fundamental Theorem of Calculus

What happened when a student in an AP Calculus class in a small school in rural Georgia discovered a way to bypass some laborious computations

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My favorite teaching assignments are those that involve students whose mathematical abilities exceed my own. Usually, my experience permits me to stay a step or two ahead of such students, but their observations and questions—borne of creativity and innate ability and coupled with youthful curiosity and mathematical intuition—occasionally lead me, and them, into uncharted waters. Such was the genesis of Cosby's rule. This is the story of what Cosby's rule is, how it came to be, and the effect that it had on an AP Calculus class in a small school in rural Georgia.

Washington-Wilkes Comprehensive High School is on a 4×4 block schedule. Calculus offered during the first semester includes students who intend to take the AP (AB) examination in the spring as well as students who want to take just one semester of calculus to prepare for college-level mathematics. The mathematics program at WWCHS is solid, and the students who register for calculus are well prepared. During the 2007–8 school year, the first-semester class was small, as many calculus classes are, but the second-semester class, as a result of the exodus of the non-AP candidates, numbered just four students.

Ian and Ashley had been active participants during the first semester—every assignment would prompt them to ask substantial questions about alternative methods and underlying theory—and I often used their questions to gauge the level of understanding of their classmates. Many were less inclined to ask questions but often commented, “Yeah, I didn’t understand that, either,” and then would become quite involved in subsequent discussions. John volunteered only occasionally in class but seemed to be totally comfortable with most of the material, as his outstanding performance on every assessment confirmed. Josh, whose mathematical talents were revealed only in his work, had remained quiet during most of the first semester, rarely saying a word unless specifically called on and even then saying as little as possible. For the most part, John and Josh had been disengaged throughout the first semester.

All this changed when the second semester began. The small class size provided an expanded comfort zone in which all these students could feel safe in their pursuit of understanding. John and Josh became active participants in the class, matching Ian and Ashley in both quality and quantity of observations and questions.

When justifying their work, my students knew that if their justification included faulty mathematics, and if we were pressed for time, and if the error was fairly commonplace, I, like many teachers, might just say, “That won’t work because …” and point out the error. They also knew that I rarely responded in that way. I preferred to have them assess the situation themselves, answering one another’s questions and exploring one another’s ideas through classroom discourse.

In our small group, the students quickly lost their inhibitions about asking and answering questions and suggesting ideas that we could explore. A daily ritual soon evolved in which all four, now unabashed in their explanations of their work, could consider an assortment of ideas. Of course, such conversations usually centered on applications—how they interpreted the problem, how they set it up, which methods they used to find solutions, and so forth. Only infrequently would a student come up with a new idea for an algorithmic procedure.

Students at all levels often generate ideas that are not obviously right or wrong, and, when they do, it is productive to take time to explore the ideas as a group. Some of these ideas are really strange. Occasionally, they are so strange that even the teacher might not know what to think of them.

Such was the case when I asked Josh how he had worked a problem from our textbook. First-semester calculus topics include limits and differentiation, while second semester begins with an introduction to integration. Following the textbook chapter, the students worked first with antiderivatives and basic integration rules and then were introduced to summation notation, summation formulas, the area of a plane region, upper and lower sums, and finding the area by the limit definition.

RIEMANN SUMS FOR AREA

The problem in question was no. 51 from *Calculus of a Single Variable* (Larson, Hostetler, and Edwards 2006, sec. 4.2, p. 269):

> Use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval: $y = 16 - x^2$ on $[1, 3]$

A more or less traditional solution follows.

Create an n-partition of the interval $[1, 3]$; then the size of each part is $\Delta x = (2/n)$. We exclude the function on the ith subinterval using

$$m_i = 1 + i\left(\frac{2}{n}\right),$$

for $i = 1$ to n. Then we have the following:

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[16-\left(1+\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[16-\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\right]\left(\frac{2}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[15-\frac{4i}{n}-\frac{4i^2}{n^2}\right]\left(\frac{2}{n}\right) && (1)\\
&= \lim_{n\to\infty}\left[30\left(\frac{\sum_{i=1}^{n}1}{n}\right) - 8\left(\frac{\sum_{i=1}^{n}i}{n^2}\right) - 8\left(\frac{\sum_{i=1}^{n}i^2}{n^3}\right)\right] && (2)\\
&= \lim_{n\to\infty}\left[\frac{30n}{n}-\frac{8(n)(n+1)}{2n^2}-\frac{8(n)(n+1)(2n+1)}{6n^3}\right]^*\\
&= \lim_{n\to\infty}\left[\frac{30n}{n}-\frac{8n^2+8n}{2n^2}-\frac{16n^3+24n^2+8n}{6n^3}\right]^*\\
&= 30(1) - 8\left(\frac{1}{2}\right) - 8\left(\frac{1}{3}\right)\\
&= \frac{70}{3}
\end{aligned}$$

When asked how he had worked the problem, Josh said that he had taken the coefficient of each numerator, divided by the exponent in the denominator, and added those results. He had noticed a pattern in the examples and explained further: "I got to the summation notation (1), and then I just sort of mentally multiplied through by the $(2/n)(2)$. I divided the reduced number in the numerator by the exponent of n in the denominator. Then I added all of that together. It worked on the first four problems, but that's all I did." Josh had completely omitted the steps marked with an asterisk in the solution above, whereas the other members of the class had gone through these or similar traditional calculations.

Although the other members of the class were skeptical about Josh's method, we dubbed his approach Cosby's rule and set out to determine whether it seemed to work all the time, some of the time, occasionally, or just on those first four problems. I had never seen such a shortcut, but I encouraged the class to keep comparing the traditional approach with Cosby's rule. The students looked back through the assignment, checking their calculations against the proposed shortcut, working any problems as yet undone, and marveling at the fact that Cosby's rule would have worked for every problem. Then they intently worked through problems that had not been assigned and discovered that the rule continued to hold.

The students enjoyed the investigation and benefitted from practicing the summation operations. In addition, the class came alive with thoughtful and excited mathematical conversation. As the investigation continued into the next textbook topics—Riemann sums and indefinite integrals—they were convinced that Josh had found *something.* But no one was ready to say exactly what that was.

HOW COSBY'S RULE WORKS

This seemed to be a good time to explain how Cosby's rule worked. To do so, we needed to see a connection between the summation notation, the definition of Riemann sums, the definition of the definite integral, and the fundamental theorem of calculus (addressed in the next section of the book).

Josh did not ignore the mathematical symbol for summation and the i notation; rather, he recognized the effect of the summation operations. The key to Cosby's rule lies in the denominators. To show how the rule works, we need to prove the following proposition:

Cosby's rule: For each positive integer k, we have

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^k}{n^{k+1}}=\frac{1}{k+1}.$$

Students often generate ideas that are not obviously right or wrong. Take time to explore these ideas as a group.

Most mathematics books provide summation formulas for $\sum i$, $\sum i^2$, and $\sum i^3$ to use in the limit formulas, and we can use those formulas to show how the formula given in the proposition above holds for $k = 0$, 1, and 2.

For $k = 0$:

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^0}{n}=\lim_{n\to\infty}\frac{\sum_{i=1}^{n} 1}{n}=\lim_{n\to\infty}\frac{n}{n}=1$$

For $k = 1$:

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^1}{n^2}=\lim_{n\to\infty}\frac{n(n+1)}{2n^2}=\frac{1}{2}$$

For $k = 2$:

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^2}{n^3}=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}=\frac{2}{6}=\frac{1}{3}$$

Taking advantage of this information, we can understand how Josh was able to omit the asterisked steps, but there are usually no textbook formulas when $k > 3$, and formulas for these sums are messy to state and derive. Fortunately, we do not need an explicit formula but can use telescoping sums to establish the limit given in Cosby's rule.

Before tackling the general case, we illustrate the argument for $k \le 3$.

For $k = 1$:

$$\begin{aligned} n^{k+1}=n^2&=\left(1^2-0^2\right)+\left(2^2-1^2\right)+\left(3^2-2^2\right)+\cdots+\left(n^2-(n-1)^2\right)\\ &=\sum_{i=1}^{n}\left(i^2-(i-1)^2\right)\\ &=\sum_{i=1}^{n}(2i-1)\\ &=2\sum_{i=1}^{n} i-\sum_{i=1}^{n}1 \end{aligned}$$

Then,

$$2\sum_{i=1}^{n} i=n^2+n,$$

so

$$\frac{\sum_{i=1}^{n} i}{n^2}=\frac{1}{2}+\frac{1}{2n}$$

and, finally,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i}{n^2}=\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)=\frac{1}{2}.$$

For $k = 2$:

$$\begin{aligned}n^{k+1}=n^3&=\left(1^3-0^3\right)+\left(2^3-1^3\right)+\left(3^3-2^3\right)\\&\quad+\cdots+\left(n^3-\left(n-1\right)^3\right)\\&=\sum_{i=1}^{n}\left(i^3-\left(i-1\right)^3\right)\\&=\sum_{i=1}^{n}\left(3i^2-3i+1\right)\\&=3\sum_{i=1}^{n}i^2-3\sum_{i=1}^{n}i+\sum_{i=1}^{n}1\end{aligned}$$

Then, dividing both sides by $3n^3$, we get

$$\frac{1}{3}=\frac{\sum_{i=1}^{n}i^2}{n^3}-\frac{\sum_{i=1}^{n}i}{n^3}+\frac{\sum_{i=1}^{n}1}{3n^3}.$$

Because

$$\sum_{i=1}^{n}i\le n^2 \text{ and } \sum_{i=1}^{n}1\le n,$$

when we take the limit as $n\to\infty$, we have the following:

$$\begin{aligned}\frac{1}{3}&=\lim_{n\to\infty}\frac{\sum_{i=1}^{n}i^2}{n^3}-\lim_{n\to\infty}\frac{\sum_{i=1}^{n}i}{n^3}+\lim_{n\to\infty}\frac{\sum_{i=1}^{n}1}{3n^3}\\\frac{1}{3}&=\lim_{n\to\infty}\frac{\sum_{i=1}^{n}i^2}{n^3}-0+0\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}i^2}{n^3}=\frac{1}{3}$$

is in agreement with Cosby's rule.

For $k = 3$:

$$\begin{aligned}n^{k+1}=n^4&=\left(1^4-0^4\right)+\left(2^4-1^4\right)+\left(3^4-2^4\right)\\&\quad+\cdots+\left(n^4-\left(n-1\right)^4\right)\\&=\sum_{i=1}^{n}\left(i^4-\left(i-1\right)^4\right)\\&=\sum_{i=1}^{n}4i^3+\sum_{i=1}^{n}\left(-6i^2+4i-1\right)\end{aligned}$$

Dividing by $4n^4$, we obtain the following:

$$\frac{1}{4}=\frac{\sum_{i=1}^{n}4i^3}{n^4}+\frac{\sum_{i=1}^{n}\left(-6i^2+4i-1\right)}{4n^4}$$

Note that

$$\sum_{i=1}^{n}i^k\le n\bullet n^k=n^{k+1}.$$

So

$$\left|\sum_{i=1}^{n}\frac{\left(-6i^2+4i-1\right)}{4n^4}\right|\le\frac{6n^3+4n^2+n}{4n^4},$$

and, by the squeeze principle,

$$\frac{\sum_{i=1}^{n}\left(-6i^2+4i-1\right)}{4n^4}=0.$$

Therefore,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}4i^3}{n^4}=\frac{1}{4}$$

is in agreement with Cosby's rule.

For general k:
For general k, then, we can follow this same path and see that

$$\begin{aligned}n^{k+1}&=\sum_{i=1}^{n}i^{k+1}-\left(i-1\right)^{k+1}\\&=\sum_{i=1}^{n}\left(i^k\right)\left(k+1\right)+\sum_{i=1}^{n}p(i),\end{aligned}$$

where p is a polynomial of degree $k - 1$. Dividing by $(n^{k+1})(k + 1)$, we get

$$\frac{1}{k+1}=\frac{\sum_{i=1}^{n}i^k}{n^{k+1}}+\frac{\sum_{i=1}^{n}p(i)}{n^{k+1}}.$$

Taking the limit as $n\to\infty$, we obtain the full force of Cosby's rule:

$$\frac{1}{k+1}=\lim_{n\to\infty}\frac{\sum_{i}^{n}i^k}{n^{k+1}}$$

Let f be defined on the closed interval [0, 1]. For each positive integer n, take $\Delta x = 1/n$, and $c_i = i/n$ for $i = 1, \ldots, n$.

Then the sum

$$R_n=\sum_{i=1}^{n}f\left(c_i\right)\Delta x$$

is a Riemann sum. If

$$\lim_{n \to \infty} R_n$$

exists, this limit is called the definite integral of f from 0 to 1, denoted as

$$\int_0^1 f(x)\,dx.$$

Applying this to $f(x) = x^k$ on the interval [0, 1], we obtain the following:

$$R_n = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^k \left(\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{i^k}{n^{k+1}}$$

Therefore,

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^k}{n^{k+1}}.$$

Saying that

$$\int_0^1 x^k = \frac{1}{k+1}$$

is the same as saying that

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^k}{n^{k+1}} = \frac{1}{k+1}.$$

This is precisely the limit Josh evaluated, and the result of Cosby's rule is equivalent to evaluating the definite integral:

$$\int_0^1 x^k\,dx = \frac{1}{k+1}$$

WHAT JOSH DISCOVERED

There are two equally valid ways of looking at Cosby's rule: (1) Josh intuited a special case of the fundamental theorem of calculus; or (2) once we have the fundamental theorem, we can get a quick proof of Cosby's rule.

My students did not have enough experience with these symbols to articulate—much less prove—what Josh had discovered. Initially, they had never heard of indefinite integrals or the fundamental theorem of calculus. But as they continued to work through the investigation and became immersed in these concepts, their understanding of new topics was better assimilated. Their transition from summation operations to integration and the fundamental theorem of calculus was clearly more complete than had they not explored Cosby's rule.

My students' curiosity about the validity and utility of Cosby's rule took them places I could never have taken them alone, and their excitement transcended the classroom itself. Within days, they had discussed Cosby's rule with the first-semester class members and with the other mathematics teachers. One student even called his girlfriend, who was taking the BC course in an Arizona school, to see if she could figure it out. The whole school was abuzz about Cosby's rule.

Students' curiosity about Cosby's rule took them places I could never have taken them alone.

As often as possible, I try to capitalize on students' ideas to promote discourse and understanding. Usually, within a class period, students can be guided toward a confirmation of a classmate's conjecture or show it to be false. In this case, however, I admit that I did not know how to explain Cosby's rule fully. Using a classroom discovery approach was as much for my benefit as for my students'.

Especially exciting for the students was to be a part of the process—to see how definitions, rules, and theorems of mathematics might be developed over time, as when someone notices a pattern or unusual result and tries to prove that something is true. For me, this investigation was a reminder not to dismiss students' observations as simple coincidences but to use those observations to stimulate students' curiosity and enthusiasm. In this way they build deeper and more meaningful mathematical understanding.

ACKNOWLEDGMENTS

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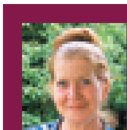
REFERENCE

Larson, Ron, Robert P. Hostetler, and Bruce H. Edwards. *Calculus of a Single Variable.* 8th ed. New York: Houghton Mifflin Co., 2006. See chap. 4, "Integration."



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